Lunar Radio Beacon Location by Doppler Measurements

TAYLOR GABBARD* AND R. M. L. BAKER JR.†
Lockheed-California Company, West Los Angeles, Calif.

A feasibility analysis points up the fact that, although the overall Doppler shift of a signal originating from a beacon on the moon is quite significant, the portion of the shift which is sensitive to the selenographic latitude and longitude of the beacon is much smaller. In fact, the basic transmitter frequency would have to be held constant (including noise) to one part in 10^9 for even marginal performance of the beacon location system. The feasibility analysis also shows, qualitatively, that there exist optimum times of the month for most accurately determining the selenographic coordinates of a beacon. The detailed formalism for a differential correction procedure is given in which the first-estimate values of the beacon coordinates are obtained, e.g., from the trajectory ephemeris of the vehicle setting the beacon upon the moon.

I. Introduction

THE question considered here is that of determining by analysis of the Doppler shift in frequency, as measured from a terrestrial observing station, the selenographic coordinates of a radio beacon set upon the moon. It is recognized that the sensitivities required here may be pushing the present state of the art in Doppler measurement; however, the analysis is continued under the assumption that the reduction of the raw Doppler data will lead to the more directly usable range-rate ($\dot{\rho}$) information that the authors desire.

The analysis presented here is divided into two parts: first, recognizing the obvious, that $\dot{\rho}$ will arise from the rotational motion of the earth and moon about their axes and the motion of revolution of the moon about the earth, a rough, rule-of-thumb analysis is carried out to estimate the relative contributions of these motions to the measured $\dot{\rho}$ and to determine whether certain periods of the month are more favorable than others for accurately determining the location of the lunar beacon. A more exact and more detailed analysis is developed next which in essence is a differential correction procedure based upon $\dot{\rho}$ as the observable quantity. The first-estimate values for the selenographic coordinates of the beacon may be obtained, e.g., from knowledge of the trajectory ephemeris of the vehicle planting the beacon on the moon and the time at which the beacon was placed there.

II First-Order Analysis

It was noted in the Introduction that $\dot{\rho}$ arises from the rotational motion of the earth and moon and from the motion of revolution of the moon about the earth. To obtain a somewhat more detailed knowledge beyond this general statement, first note that

$$\boldsymbol{\varrho} = \mathbf{R} + \mathbf{r} + \boldsymbol{\mathfrak{R}} \tag{1}$$

(The symbols are defined in Fig. 1.) Differentiation of Eq. (1) leads to

$$\dot{\boldsymbol{\varrho}} = \dot{\mathbf{R}} + \dot{\mathbf{r}} + \dot{\boldsymbol{\mathfrak{R}}} \tag{2}$$

and Eqs. (1) and (2) in turn lead to

$$\rho\dot{\rho} = \boldsymbol{\varrho}\cdot\dot{\boldsymbol{\varrho}} = \{\dot{\mathbf{R}}\cdot(\mathbf{R} + \mathbf{r}) + \mathbf{R}\cdot\dot{\mathbf{r}} + [\dot{\mathbf{R}}\cdot\boldsymbol{\mathfrak{R}} + \mathbf{R}\cdot\dot{\boldsymbol{\mathfrak{R}}}]\} + (\dot{\mathbf{r}} + \dot{\boldsymbol{\mathfrak{R}}})\cdot(\mathbf{r} + \boldsymbol{\mathfrak{R}}) \tag{3}$$

or

$$\rho\dot{\rho} \triangleq \{(\rho\dot{\rho})_1 + (\rho\dot{\rho})_2\} + (\rho\dot{\rho})_3 \tag{4}‡$$

where the associations are clear.

The terms in braces in Eq. (3) are recognized as those which arise from the fact that the terrestrial observing station is not located at the earth's center, and the brackets further set apart those terms dependent upon the lunar coordinates of the beacon. Next consider, in turn, $\dot{\rho}_1$, $\dot{\rho}_2$, and $\dot{\rho}_3$, i.e., the respective contributions to $\dot{\rho}$ from $(\rho\dot{\rho})_1$, $(\rho\dot{\rho})_2$, and $(\rho\dot{\rho})_3$.

Contribution of $(\rho\dot{\rho})_1$ to $\dot{\rho}$

It has been noted that $(\rho\dot{\rho})_1$ is independent of the selenographic coordinates of the lunar beacon; hence $\dot{\rho}_1$ resulting from it is a masking, superfluous signal that will yield no knowledge of the beacon's lunar coordinates. To obtain a maximum order-of-magnitude value for it, make the simplifying assumptions that the Doppler station is on the equator, that the earth's equator and the lunar plane of motion are coincident, and that $\rho \approx r$. By these assumptions,

$$(\rho\dot{\rho})_1 \approx \dot{\mathbf{R}}\cdot\boldsymbol{\varrho} + \mathbf{R}\cdot\dot{\mathbf{r}} \tag{5}$$

hence

$$\dot{\rho}_1 \triangleq (\rho\dot{\rho})_1/\rho \approx \dot{\mathbf{R}}\cdot(\boldsymbol{\varrho}/\rho) + (\mathbf{R}/\rho)\cdot\dot{\mathbf{r}} \tag{6}$$

Closer examination of Eq. (6) reveals that the first term is maximum positive when the moon is on the western horizon, i.e., when it is setting, and that the second term is maximum positive when the moon is on the eastern horizon, i.e., when it is rising and when the moon is at perigee. Since the two conditions of moonrise and moonset cannot be met simultaneously, the relative magnitudes of the two terms in Eq. (6) must be examined:

$$[\dot{\mathbf{R}}\cdot(\boldsymbol{\varrho}/\rho)]_{\max} \approx 5.9 \times 10^{-2} \text{ c.u.}§ \tag{7}$$

and, where $\rho \approx r \approx 60$ c.u.,

$$[(\mathbf{R}/\rho)\cdot\dot{\mathbf{r}}]_{\max} \approx 2.2 \times 10^{-3} \text{ c.u.} \tag{8}$$

Combining Eqs. (7) and (8) into Eq. (6),

$$(\dot{\rho}_1)_{\max} \approx 5.7 \times 10^{-2} \text{ c.u.}$$
$$\approx 1470 \text{ fps} \tag{9}$$

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* Senior Scientist, Astrodynamics Research Center.
† Head of Astrodynamics Research Center; also Assistant Professor of Engineering, University of California, Los Angeles, Calif.

‡ The symbol $\triangleq$ is to be read as "is equal by definition to."

§ The set of units termed "characteristic units" (and hereafter abbreviated simply as c.u.) is based upon the earth's equatorial radius (a_e) as the unit of distance and the "k_e^{-1} min" (approximately 13.5 min) as the unit of time; $k_e^2 = GM_E$, where (in laboratory units) G is the constant of universal gravitation and M_E is the mass of the earth. In characteristic units, the mass of the earth is taken as the mass unit.

Contribution of $(\rho\dot{\rho})_2$ to $\dot{\rho}$

Since the position of the observing station is involved in $(\rho\dot{\rho})_2$, the value of $\dot{\rho}_2$ also will have a principal diurnal variation. Again, to make order-of-magnitude estimates of the quantities involved, the authors make some simplifying assumptions. Noting that the lunar equator is not inclined too greatly to the earth's equator (approximately 25° at present), the geometry is simplified by assuming that the earth's equator, the moon's equator, and the moon's orbital plane are coincident and that the observing station and beacon lie on the equators of the earth and moon, respectively.

Under the simplifying assumptions given, $\dot{\mathbf{R}}\cdot\mathfrak{R}$ will be a maximum when the moon is in the observer's meridian and the beacon is on the west limb of the moon, i.e., at large positive longitude. Similarly, $\mathbf{R}\cdot\dot{\mathfrak{R}}$ will be a maximum when the observer sees the moon setting and when the beacon is in the center of the lunar disk. Again one finds conditions that cannot be met simultaneously; to determine the principal effect, a comparison of the relative magnitudes of the two quantities concernedmust be made next; hence

$$[\dot{\mathbf{R}}\cdot(\mathfrak{R}/\rho)]_{\max} \approx 2.5 \times 10^{-4}\ \text{c.u.} \tag{10}$$

and

$$[(\mathbf{R}/\rho)\cdot\dot{\mathfrak{R}}]_{\max} \approx 8.8 \times 10^{-6}\ \text{c.u.} \tag{11}$$

where it is taken that $\rho \approx r \approx 60$ c.u. and $\mathfrak{R} \approx \frac{1}{4}$ c.u. Since the latter is approximately two orders of magnitude down from the former,

$$\dot{\rho}_2 \triangleq (\boldsymbol{\rho}\cdot\dot{\boldsymbol{\rho}})_2/\rho \tag{12}$$

has a maximum daily variation of

$$(\dot{\rho}_2)_{\max} \approx 2.5 \times 10^{-4}\ \text{c.u.}$$
$$\approx 7.0\ \text{fps} \tag{13}$$

It is evident that $\dot{\rho}_2$ would diminish if either the beacon or the observing station were situated at higher latitudes. Thus, using the criterion that the selenographic coordinates of the beacon will be determined more accurately for those positions yielding the greatest contribution to the measured $\dot{\rho}$, it is seen that the component $\dot{\rho}_2$ would be expected to yield more accurate data for longitudes near the limb and for low latitudes.

Contribution of $(\rho\dot{\rho})_3$ to $\dot{\rho}$

It is recognized that $(\rho\dot{\rho})_3$ is exactly identical to $\rho\dot{\rho}$ for the hypothetical assumption in which the observing Doppler station is located at the earth's center. Imagining this assumption fulfilled, one sees from Fig. 1 that by the law of cosines

$$\rho^2 = r^2 + \mathfrak{R}^2 - 2r\mathfrak{R}\cos d \tag{14}$$

where

$$\cos d = \sin b \sin\beta + \cos b \cos\beta \cos(\lambda - l) \tag{15}$$

The selenographic latitude-longitude of the beacon and sub-earth point, respectively, are (λ, β) and (l, b); λ and β are, in part, the quantities that one wishes to determine from the Doppler observations, and l and b are quantities that may be computed for any instant of time. (They also are tabulated in the *American Ephemeris and Nautical Almanac* at one-day intervals.) Differentiating Eq. (14), one finds that

$$(\rho\dot{\rho})_3 \approx r\dot{r} - \dot{r}\mathfrak{R}[\sin b \sin\beta + \cos b \cos\beta \cos(\lambda - l)] - r\mathfrak{R}[\dot{b}\cos b \sin\beta - \dot{b}\sin b \cos\beta \cos(\lambda - l) + \dot{l}\cos b \cos\beta \sin(\lambda - l)] \tag{16}$$

Since l and b lie in the general range

$$-8° < l \qquad b < +8° \tag{17}$$

one makes the further assumption that they are zero and obtains

$$(\rho\dot{\rho})_3 \approx r\dot{r} - r\mathfrak{R}\dot{b}\sin\beta - \dot{r}\mathfrak{R}\cos\beta\cos\lambda - r\mathfrak{R}\dot{l}\cos\beta\sin\lambda \tag{18}$$

With a little thought, it is seen that both $\dot{l}$ and $\dot{b}$ have their maximum values in the vicinity of the ecliptic nodal passages of the moon. More exactly, $\dot{l}, \dot{b} < 0$ (both $\approx -2.25 \times 10^{-4}$ c.u.) at the ascending node, and $\dot{l}, \dot{b} > 0$ (both $\approx 2.50 \times 10^{-4}$ c.u.) at the descending node.

Looking at the individual components of Eq. (18), it is first recognized that the $(r\,\dot{r})$ term contributes no knowledge of λ and β. The maximum magnitude of its contribution to

$$\dot{\rho}_3 \triangleq (\rho\dot{\rho})_3/\rho \tag{19}$$

is[1]

$$(r\dot{r}/\rho)_{\max} \approx \dot{r}_{\max} = (\mu/p)^{1/2}\,e \tag{20}$$

and at a true anomaly $v = 90°$.

Taking the semilatus rectum $p \approx 60$ c.u. and $e \approx 0.05$ for the lunar orbit, there results

$$\dot{r}_{\max} \approx 6.4 \times 10^{-3}\ \text{c.u.}$$
$$\approx 170\ \text{fps} \tag{21}$$

Under the continued assumption that $\rho \approx r \approx 60$ c.u. and $\mathfrak{R} \approx \frac{1}{4}$ c.u., the coefficients of $\sin\beta$, $\cos\beta\cos\lambda$, and $\cos\beta\sin\lambda$ have the representative values of

$$(r\mathfrak{R}\dot{b}/\rho)_{\max} \approx 5.9 \times 10^{-5}\ \text{c.u.} \approx 1.5\ \text{fps} \tag{22}$$

$$(\dot{r}\mathfrak{R}/\rho)_{\max} \approx 2.7 \times 10^{-5}\ \text{c.u.} \approx 0.7\ \text{fps} \tag{23}$$

$$(r\mathfrak{R}\dot{l}/\rho)_{\max} \approx 5.9 \times 10^{-5}\ \text{c.u.} \approx 1.5\ \text{fps} \tag{24}$$

Again using the criterion that those locations will be determined most accurately which produce the greatest contribu-

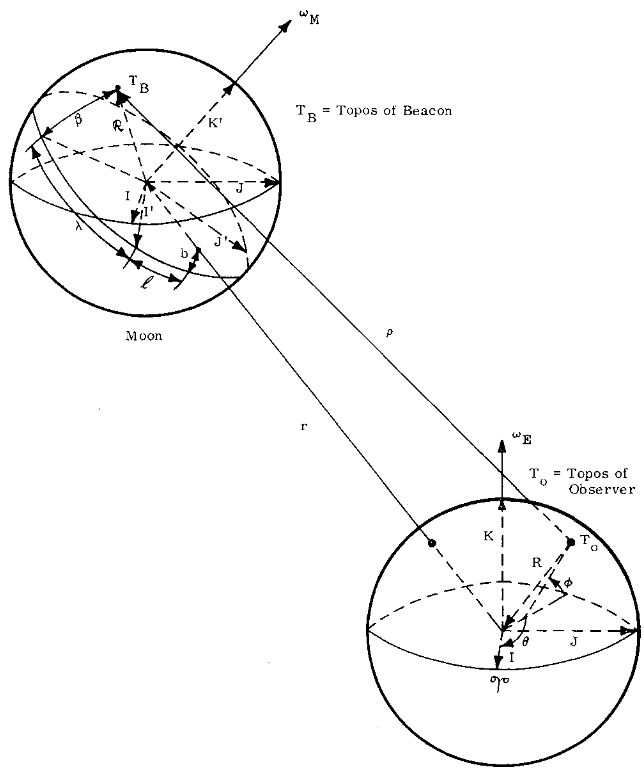

Fig. 1 Earth-moon geometry

[1] Baker, R. M. L., Jr. and Makemson, M. W., *An Introduction to Astrodynamics* (Academic Press Inc., New York 1960), pp. 115, 66–67.

tion to the measured $\dot{\rho}$, one may conclude that the libration-induced $\dot{\rho}$ will permit high accuracy in latitude β only at the sacrifice of accuracy in longitude λ, and vice versa. Examination of Eq. (18) shows that both β and λ may best be determined when the moon is at the ecliptic nodal passage, and that λ may be determined better still when this instant corresponds to a true anomaly $v = 90°$.

Thus, one has seen in this rough analysis that there are two principal sources of contribution to the measured $\dot{\rho}$, part of which lead to knowledge of the selenographic coordinates of the moon beacon. They are as follows:

1) The rotation of the earth (the effect of the moon's rotation is of second order yet to this).

2) The motion of revolution of the moon about the earth through a) the varying $\dot{r}$ component, due to the nonzero eccentricity of the lunar orbit, and b) the lunar librations, due mainly to the nonzero eccentricity of the lunar orbit and the inclination of the lunar equator to the lunar orbit plane.

Based on what has gone before, the following conclusions can be drawn:

1) The influence of beacon location on the moon changes the Doppler shift pattern only a few feet per second, at most. (This location-dependent shift is approximately three orders of magnitude down from the total expected Doppler signal.) In order to sense this few feet per second, the frequency of both the terrestrial transmitter and the beacon transponder must have a fidelity of one part in 10^9.‖

2) The definition of beacon longitude will generally be somewhat better than the determination of beacon latitude.

3) Beacon latitude can be better determined when the moon is a) on the observing station's horizon, and b) at the ecliptic nodal passages.

4) Beacon longitude can be better determined when the moon is a) on the observing station's horizon, b) at the ecliptic nodal passage, and c) at a true anomaly of 90°.

III. Detailed Analysis

In the Introduction, it was stated that the approach here would be one of a differential correction procedure based upon $\dot{\rho}$ as the observable quantity. This procedure is one of a number of statistical methods that might be used to separate the very small Doppler shift frequency due to lunar station location from the noise. To explain this more fully, first define X_1', Y_1', and Z_1' as the first-guess estimate of the selenographic coordinates of the lunar beacon. Based upon these, one then may obtain (by formulas to be developed later) computed values of $\boldsymbol{\varrho}$ and $\dot{\rho}$ (i.e., of $\boldsymbol{\varrho}_c$ and $\dot{\rho}_c$) for any instant of time; $\boldsymbol{\varrho}_c$ and $\dot{\rho}_c$ then combine to obtain the computed $\dot{\rho}_c$:

$$\dot{\rho}_c = (\rho_c\dot{\rho}_c/\rho_c) = [(\boldsymbol{\varrho}_c\cdot\dot{\boldsymbol{\varrho}}_c)/(\boldsymbol{\varrho}_c\cdot\boldsymbol{\varrho}_c)^{1/2}] \tag{25}$$

Using a linearized Taylor's expansion, one next has

$$(\dot{\rho}_0 - \dot{\rho}_c) \triangleq \Delta\dot{\rho} = (\partial\dot{\rho}/\partial X')\Delta X' + (\partial\dot{\rho}/\partial Y')\Delta Y' + (\partial\dot{\rho}/\partial Z')\Delta Z' \tag{26}$$

where $\dot{\rho}_0$ is the *observed* value of $\dot{\rho}$ corresponding to the same instant of time for which $\dot{\rho}_c$ was obtained. The partial derivative coefficients to $\Delta X'$, $\Delta Y'$, and $\Delta Z'$ are obtained from the same formulas used in obtaining $\dot{\rho}_c$.

Thus, with a minimum of three "observed-minus-computed" residuals [i.e., three of Eq. (26)], one may solve for corrections to the first-guess values of the selenographic coordinates and obtain the improved values

$$\begin{aligned} X_2' &= X_1' + \Delta X' \\ Y_2' &= Y_1' + \Delta Y' \\ Z_2' &= Z_1' + \Delta Z' \end{aligned} \tag{27}$$

‖ That is, for the signals leading to knowledge of the beacon location, $\Delta\nu/\nu = v/c \approx 0.7 \text{ fps}/0.98 \times 10^9 \text{ fps} \approx 10^{-9}$.

Iterating on the procedure, the values given by Eqs. (27) could be used again in Eqs. (25) and (26) to get further improved values of the beacon coordinates.

Proposed herein has been a description of the basic method of differential correction; however, for the immediate problem, it is felt best to "subtract-out" of $\dot{\rho}_0$ that computed portion of $\dot{\rho}$ which does not depend upon the lunar beacon location and to use for $\dot{\rho}_c$ just those portions of $\dot{\rho}$ which do depend upon the lunar location of the beacon. Necessarily, the coefficients in Eq. (26) must be based upon only the latter, i.e., upon the portion of $\dot{\rho}$ dependent upon beacon location. To see more clearly the point to be made, rewrite Eq. (3) in the manner

$$\rho_c\dot{\rho}_c = [(\dot{\mathbf{R}}_c + \dot{\mathbf{r}}_c)\cdot(\mathbf{R}_c + \mathbf{r}_c)] + [\dot{\mathfrak{R}}_c\cdot\boldsymbol{\varrho}_c + \mathfrak{R}_c\cdot(\dot{\mathbf{R}}_c + \dot{\mathbf{r}}_c)] \tag{28}$$

Now, replace $\dot{\rho}_0$ and $\dot{\rho}_c$ in Eq. (26) by

$$\dot{\rho}_0' = \dot{\rho}_0 - (1/\rho_c)[(\dot{\mathbf{R}}_c + \dot{\mathbf{r}}_c)\cdot(\mathbf{R}_c + \mathbf{r}_c)] \tag{29}$$

$$\dot{\rho}_c' = (1/\rho_c)[\dot{\mathfrak{R}}_c\cdot\rho_c + \mathfrak{R}_c\cdot(\dot{\mathbf{R}}_c + \dot{\mathbf{r}}_c)] \tag{30}$$

and take Eq. (26) replaced by

$$(\dot{\rho}_0' - \dot{\rho}_c') = (\partial\dot{\rho}'/\partial X')\Delta X' + (\partial\dot{\rho}'/\partial Y')\Delta Y' + (\partial\dot{\rho}'/\partial Z')\Delta Z' \tag{26a}$$

where the partial differential coefficients now are obtained from

$$\dot{\rho}' = (1/\rho)[\dot{\mathfrak{R}}\cdot\boldsymbol{\varrho} + \mathfrak{R}\cdot(\dot{\mathbf{R}} + \dot{\mathbf{r}})] \tag{30a}$$

Turning now to the development of more explicit expressions, first define the frame of reference as the true (celestial) equator and equinox of date. Defining unit vectors in this frame as $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$ (see Fig. 1), one may write (see footnote 1, pp. 66–67)

$$\begin{aligned} \mathbf{R} = &-(C + H)\cos\varphi\cos\theta\,\mathbf{I} \\ &-(C + H)\cos\varphi\sin\theta\,\mathbf{J} \\ &-(S + H)\sin\varphi\,\mathbf{K} \end{aligned} \tag{31}$$

and

$$\dot{\mathbf{R}} = \omega_E(C + H)\cos\varphi\sin\theta\,\mathbf{I} - \omega_E(C + H)\cos\varphi\cos\theta\,\mathbf{J} \tag{32}$$

where (see Fig. 1) φ is the geodetic latitude of the Doppler observing station, θ is the local true sidereal time at the station, and $\omega_E \approx \dot{\theta}$ is the scalar angular rate of rotation of the earth; H is the height of the observing station above the reference spheroid, and C and S are functions of φ and the flattening f of the reference spheroid.

For the geocentric position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ of the moon, one simply writes

$$\mathbf{r} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K} \tag{33}$$

and, to a sufficient approximation,

$$\dot{\mathbf{r}} = \dot{x}\mathbf{I} + \dot{y}\mathbf{J} + \dot{z}\mathbf{K} \tag{34}$$

For the position $\mathfrak{R}$ and velocity $\dot{\mathfrak{R}}$ of the beacon in the true (celestial) equator and equinox of date frame, which has been translated to be selenocentered, one writes

$$\mathfrak{R} = X\mathbf{I} + Y\mathbf{J} + Z\mathbf{K} \tag{35}$$

and

$$\dot{\mathfrak{R}} = \dot{X}\mathbf{I} + \dot{Y}\mathbf{J} + \dot{Z}\mathbf{K} \tag{36}$$

where the selenocentric components of position (X,Y,Z) and of velocity $(\dot{X},\dot{Y},\dot{Z})$ are related to the selenographic components of position (X',Y',Z') by the matrix equations

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = a_{ij} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} \tag{37}$$

and

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \dot{a}_{ij} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} \tag{38}$$

In other words, the matrix a_{ij} transforms position components from the true selenographic frame (defined by Cassini's laws for the mean rotation of the moon and expressions given by Hayn for the physical librations) into the selenocentric, true (celestial) equator and equinox of date frame. The term $\dot{a}_{ij}$ is derived in a relatively simple manner from the $\dot{a}_{ij}$, and Eq. (38) contains the assumption that the beacon is at rest with respect to lunar features. The terms a_{ij} and $\dot{a}_{ij}$ may be computed for any instant of time. Next, combining Eqs. (31, 33, 35, and 37) in the manner prescribed by Eq. (1), there results

$$\begin{aligned} \boldsymbol{\rho} = {} & [-(C + H)\cos\varphi\cos\theta + x + \\ & (a_{11}X' + a_{12}Y' + a_{13}Z')]\,\mathbf{I} + \\ & [-(C + H)\cos\varphi\sin\theta + y + \\ & (a_{21}X' + a_{22}Y' + a_{23}Z')]\,\mathbf{J} + \\ & [-(S + H)\sin\varphi + z + \\ & (a_{31}X' + a_{32}Y' + a_{33}Z')]\,\mathbf{K} \end{aligned} \tag{39}$$

or, where the associations are clear,

$$\triangleq A_1\mathbf{I} + A_2\mathbf{J} + A_3\mathbf{K} \tag{39a}$$

Similarly,

$$\begin{aligned} \mathbf{R} + \mathbf{r} = {} & [-(C + H)\cos\varphi\cos\theta + x]\,\mathbf{I} + \\ & [-(C + H)\cos\varphi\sin\theta + y]\,\mathbf{J} + \\ & [-(S + H)\sin\varphi + z]\,\mathbf{K} \end{aligned} \tag{40}$$

or, again where the associations are clear,

$$\mathbf{R} + \mathbf{r} \triangleq B_1\mathbf{I} + B_2\mathbf{J} + B_3\mathbf{K} \tag{40a}$$

Also,

$$\begin{aligned} \dot{\mathbf{R}} + \dot{\mathbf{r}} = {} & [\omega_E(C + H)\cos\varphi\sin\theta + \dot{x}]\mathbf{I} + \\ & [-\omega_E(C + H)\cos\varphi\cos\theta + \dot{y}]\mathbf{J} + \dot{z}\,\mathbf{K} \end{aligned} \tag{41}$$

or

$$\dot{\mathbf{R}} + \dot{\mathbf{r}} \triangleq C_1\mathbf{I} + C_2\mathbf{J} + C_3\mathbf{K} \tag{41a}$$

Introducing the further simplifying notation of

$$\mathfrak{R} = D_1\mathbf{I} + D_2\mathbf{J} + D_3\mathbf{K} \tag{42}$$

and

$$\dot{\mathfrak{R}} = E_1\mathbf{I} + E_2\mathbf{J} + E_3\mathbf{K} \tag{43}$$

where

$$\begin{aligned} D_1 &\triangleq a_{11}X' + a_{12}Y' + a_{13}Z' \\ D_2 &\triangleq a_{21}X' + a_{22}Y' + a_{23}Z' \\ D_3 &\triangleq a_{31}X' + a_{32}Y' + a_{33}Z' \end{aligned} \tag{44}$$

and

$$\begin{aligned} E_1 &\triangleq \dot{a}_{11}X' + \dot{a}_{12}Y' + \dot{a}_{13}Z' \\ E_2 &\triangleq \dot{a}_{21}X' + \dot{a}_{22}Y' + \dot{a}_{23}Z' \\ E_3 &\triangleq \dot{a}_{31}X' + \dot{a}_{32}Y' + \dot{a}_{33}Z' \end{aligned} \tag{45}$$

one may write [see Eq. (29)]

$$\dot{\rho}_0' = \dot{\rho}_0 - \frac{B_1C_1 + B_2C_2 + B_3C_3}{(A_1^2 + A_2^2 + A_3^2)^{1/2}} \tag{46}$$

and [see Eq. (30)]

$$\dot{\rho}_c' = \frac{(A_1E_1 + A_2E_2 + A_3E_3) + (C_1D_1 + C_2D_2 + C_3D_3)}{(A_1^2 + A_2^2 + A_3^2)^{1/2}} \tag{47}$$

For some measure of simplicity in computation, note that

$$A_i = B_i + D_i \qquad i = 1, 2, 3 \tag{48}$$

and

$$\partial A_i/\partial X' = \partial D_i/\partial X' \qquad X' \rightarrow Y', Z' \tag{49}$$

One may further obtain from Eqs. (47) and (49)

$$\begin{aligned} \frac{\partial \dot{\rho}'}{\partial X'} = {} & \frac{(C_1 + E_1)(\partial D_1/\partial X') + (C_2 + E_2)(\partial D_2/\partial X') + (C_3 + E_3)(\partial D_3/\partial X')}{(A_1^2 + A_2^2 + A_3^2)^{1/2}} + \\ & \frac{A_1(\partial E_1/\partial X') + A_2(\partial E_2/\partial X') + A_3(\partial E_3/\partial X')}{(A_1^2 + A_2^2 + A_3^2)^{1/2}} - \frac{(A_1E_1 + A_2E_2 + A_3E_3)[A_1(\partial D_1/\partial X') + A_2(\partial D_2/\partial X') + A_3(\partial D_3/\partial X')]}{[(A_1^2 + A_2^2 + A_3^2)^3]^{1/2}} - \\ & \frac{(C_1D_1 + C_2D_2 + C_3D_3)[A_1(\partial D_1/\partial X') + A_2(\partial D_2/\partial X') + A_3(\partial D_3/\partial X')]}{[(A_1^2 + A_2^2 + A_3^2)^3]^{1/2}} \quad X' \rightarrow Y', Z' \end{aligned} \tag{50}$$

The partial derivatives $\partial D_i/\partial X'$ and $\partial E_i/\partial X'$ are obtained from Eqs. (44) and (45), where it is seen, e.g., that $\partial D_1/\partial X' = a_{11}$.

In conclusion, one sees that, in principle, if one is given the first-estimate values of the selenographic coordinates (X', Y', Z') of a beacon, a differential correction procedure may be carried out according to the formulas herein developed.